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List 2-facial 5-colorability of plane graphs with girth at least 12[☆]

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ABSTRACT

A proper vertex coloring of a plane graph is 2-facial if any two different vertices joined by a facial walk of length 2 are colored differently, and it is 2-distance if every two vertices at distance 2 from each other are colored differently. Note that any 2-facial coloring of a subcubic graph is 2-distance.

It is known that every plane graph with girth at least 14 has a 2-facial 5-coloring [M. Montassier, A. Raspaud, A note on 2-facial coloring of plane graphs. Inform. Process. Lett. 98 (6) (2006) 235–241], and that every planar subcubic graph with girth at least 13 has a list 2-distance 5-coloring [F. Havet, Choosability of square of planar subcubic graphs with large girth, Discrete Math. 309 (2009) 3353–3563].

We strengthen these results by proving the list 2-facial 5-colorability of plane graphs with girth at least 12.

1. Introduction

By a graph we mean a non-oriented graph without loops and multiple edges. By $V(G)$, $E(G)$, $F(G)$, $\Delta(G)$, and $g(G)$ denote the sets of vertices, edges, and faces, maximum degree, and girth of a plane graph G, respectively. (We will drop the argument when the graph is clear from context.)

A vertex coloring of a plane graph is k-*cyclic* if any two different vertices lying in the boundary of a face of size at most k are colored differently. This notion was introduced by Ore and Plummer [25]. Let σ_k be the smallest number of colors sufficient to k-cyclically color all plane graphs. We have $\sigma_3 \le 4$ by the Four Color Theorem [1] and $\sigma_4 \le 6$ [2,3]. Furthermore, Borodin et al. [15] proved that $\sigma_5 \le 8$.

Kràl et al. [23] suggested the following extension of cyclic colorings. A vertex coloring of a plane graph is 2-*facial* if any two different vertices joined by a facial walk of length at most 2 are colored differently. By φ_2 denote the smallest number of colors sufficient to color all plane graphs 2-facially. In particular, it was proved in [23] that $\varphi_2 \le 8$, which implies $\sigma_5 \le 8$, since each 2-facial coloring is obviously 5-cyclic. The general bound 8 in [23] was improved by Montassier and Raspaud [24] for plane graphs with large enough girth: $\varphi_2 \le 7$ if $g \ge 8$, $\varphi_2 \le 6$ if $g \ge 10$, and $\varphi_2 \le 5$ if $g \ge 14$. Ivanova [19] proved that $\varphi_2 \le 4$ if $g \ge 22$. Note that $\varphi_2(K_{1,3}) = 4$ while $g(K_{1,3}) = \infty$.

Now suppose each vertex v of a graph G is given a list $L(v)$ of colors. A graph G is said to be *list* 2-*facially* k-*colorable* if every list L such that $|L(v)| \ge k$ for each $v \in V(G)$ contains a 2-facial coloring.

The purpose of this paper is to improve a result in [24] as follows.

Theorem 1. *Every plane graph with girth at least* 12 *is list* 2-*facial* 5-*colorable.*

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On the other hand, 2-facial colorings are related to 2-distance colorings. A coloring of G is 2-*distance* if any two vertices at distance at most two from each other get different colors. Clearly, 2-distance colorings are usual colorings for the square of a graph. The minimum number of colors in 2-distance colorings of G is denoted by $\chi_2(G)$.

By $\chi_2^l(G)$ denote the list 2-distance chromatic number of G, i.e., the smallest k for which G is list 2-distance k-colorable. Clearly, for $\Delta = 2$ there are graphs with $\chi_2 = 4$ and arbitrarily large girth, say, C_{3k+1}. In [11,4,20,12] we give sufficient conditions (in terms of g and Δ) for χ_2^l to equal the trivial lower bound $\Delta + 1$, which can be summarized as follows.

Theorem 2. *If G is a planar graph, then $\chi_2^l = \Delta + 1$ in each of the cases $(i - vi)$:*

(i) $\Delta = 3, g \geq 24$;
(ii) $\Delta = 4, g \geq 15$;
(iii) $\Delta = 5, g \geq 12$;
(iv) $\Delta \geq 6, 10 \leq g \leq 11$;
(v) $\Delta \geq 10, 8 \leq g \leq 9$;
(vi) $\Delta \geq 16, g = 7$.

For $g \leq 6$, there are planar graphs with $\chi_2 = \Delta + 2$ for arbitrarily large Δ (see [4,16]). Still, Borodin et al. [13,14] proved that $\chi_2 = \Delta + 1$ whenever $\Delta \geq 31$ for planar graphs of girth six with the additional assumption that each edge is incident with a vertex of degree two. Dvořák et al. [16] proved that every planar graph with $\Delta \geq 8821$ and $g \geq 6$ has $\chi_2 \leq \Delta + 2$, and Borodin and Ivanova [6,7] weakened the restriction on Δ here to 18.

Special attention is paid to the 2-distance colorings of planar graphs with $\Delta = 3$ (called *subcubic*). In 1977, Wegner [4] (see also Jensen and Toft's monograph [22]) conjectured that every such graph has $\chi_2 \leq 7$. Furthermore, Dvořák et al. [17] proved that $\chi_2^l = 4$ if $g \geq 24$ (i.e., they independently obtained (i) in Theorem 2). Recently, Borodin and Ivanova [8] proved that $\chi_2 = 4$ if $g \geq 22$. Montassier and Raspaud [24] proved that $\chi_2 \leq 5$ if $g \geq 14$, which was improved by Ivanova and Solov'eva [21] to $g \geq 13$. Dvořák et al. in [17] proved that $\chi_2^l \leq 5$ if $g \geq 14$, which result was improved by Havet [18] to $g \geq 13$.

It is not hard to see that any 2-facial coloring of a subcubic graph is 2-distance, so from Theorem 1 we deduce a common strengthening of the above mentioned results from [24,21,17,18].

Theorem 3. *Every planar subcubic graph with girth at least 12 is list 2-distance 5-colorable.*

A distinctive feature of our proof of Theorem 1 is that a charge of vertices can be transferred along "feeding paths" to an unlimited distance. This kind of "global" discharging was introduced by Borodin et al. in [10] and used in [10,5,9] for improving results on oriented, circular, and acyclic colorings.

2. Proof of Theorem 1

Let G be a counterexample to Theorem 1 with the fewest edges. Clearly, G is connected and has no pendant edges. Euler's formula $|V| - |E| + |F| = 2$ can be rewritten as

$$(10|E| - 12|V|) + (2|E| - 12|F|) = -24,$$

where F is the set of faces of G. Hence,

$$\sum_{v \in V}(5d(v) - 12) + \sum_{f \in F}(r(f) - 12) = -24, \tag{1}$$

where $d(v)$ is the degree of vertex v, and $r(f)$ is the size of face f. The *charge* $\mu(v)$ of every vertex v of G is defined to be $5d(v) - 12$, while the *charge* $\mu(f)$ of every face f of G, to be $r(f) - 12$. Since the charge of every face is nonnegative, (1) implies that

$$\sum_{v \in V}(5d(v) - 12) < 0. \tag{2}$$

Note that the charge of a 2-vertex is -2, while the charge of a 3-vertex is 3, etc. We first describe some structural properties of G; then, based on these, we redistribute the charges, preserving their sum, so that all *new charges* $\mu^*(v)$ of vertices are non-negative (which will give a contradiction with (2)).

2.1. Basic properties of the minimal counterexample

By a *k-thread* we mean a path consisting of precisely k vertices of degree 2. A vertex of degree at least k or at most k is a k^+- or a k^--vertex, respectively. By a $(k_1, k_2, \ldots)$-*vertex* we mean a vertex of degree at least 3 that is incident with k_1-, k_2-, ... threads.

In what follows, by $c(v)$ we denote the color of a vertex v in a partial 2-facial coloring c of G, and $A(v)$ is the set of colors that are admissible for an uncolored vertex v, i.e., does not appear on the already colored 2-facial neighbors of v.

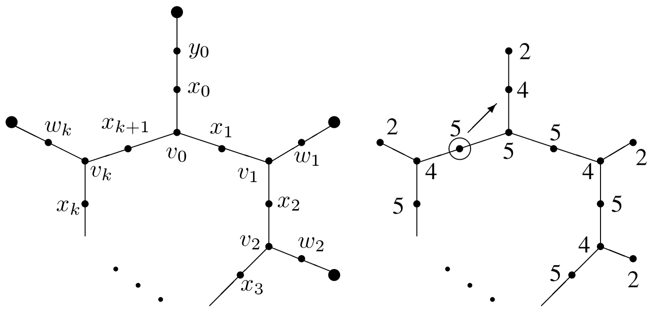

Fig. 1. The configuration in Lemma 5.

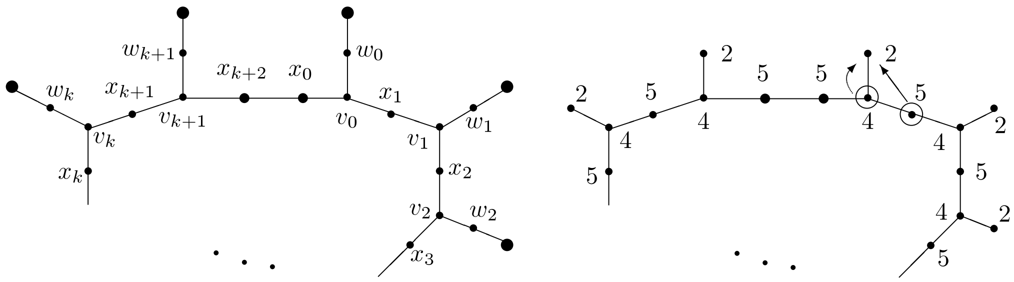

Fig. 2. The configuration in Lemma 6.

Claim 1 ([18]). *A path $x_1x_2x_3x_4$ of vertices having 2 , 2 , 3 , and 2 admissible colors, respectively, is 2-distance colorable with admissible colors.*

Claim 2 ([8]). *A path $x_1 \dots x_5$ of vertices having 2 , 2 , 3 , 3 , and 2 admissible colors, respectively, is 2-distance colorable with admissible colors.*

Lemma 3. *G has no k-thread, where $k \geq 3$.* □

Lemma 4. *G has no $(\geq 1, 2, 2)$-vertex.*

Proof. Let a $(\geq 1, 2, 2)$-vertex v be incident with paths vx_1x_2, vy_1y_2, and vz. Take a list 2-facial coloring of $G - v$ and uncolor x_1, x_2, y_1, y_2, and z. Note that $|A(y_2)| = |A(x_2)| = |A(z)| = 2$, $|A(y_1)| = |A(v)| = |A(x_1)| = 4$. Color z arbitrarily and use Claim 2. □

By a *soft path*, *SP*, we mean a path $x_1v_1x_2v_2 \dots x_kv_kx_{k+1}$, where $v_1, v_2, \dots, v_k$ are (1, 1, 1)-vertices and no v_i is joined to a v_j by a 1-thread if $|i - j| > 1$. In what follows, the 2-vertex adjacent to $v_i \in SP$ and different from x_i and x_{i+1} is denoted by w_i. Note that all w_i are pairwise distinct.

Lemma 5. *G has no* (2, 1, 1)*-vertex whose* 2*-vertices of* 1*-threads are joined by a soft path.*

Proof. Let we have a shortest cycle $C = v_0x_1v_1x_2v_2 \dots x_kv_kx_{k+1}$, where v_0 is a (2, 1, 1)-vertex incident with a 2-thread $v_0x_0y_0y_0'$, while $v_1, v_2, \dots, v_k$ are (1, 1, 1)-vertices adjacent to $w_1, w_2, \dots, w_k$ outside C, respectively (see Fig. 1). Since C is shortest, all w_i's are pairwise distinct.

Take a list 2-facial coloring of $G - C$ and uncolor w_i, where $1 \leq i \leq k$, y_0, and x_0. We can assume that $|A(v_i)| = |A(x_0)| = 4$, $|A(w_i)| = |A(y_0)| = 2$, where $1 \leq i \leq k$, and $|A(x_i)| = |A(v_0)| = 5$, where $1 \leq i \leq k + 1$. Put $c(x_{k+1}) \notin A(x_0)$. Now we can color the uncolored vertices in the following order: $w_k, v_k, x_k, w_{k-1}, v_{k-1}, x_{k-1}, \dots, w_1, v_1, x_1, v_0, y_0$, and x_0. □

Lemma 6. *G has no two* (2, 1, 1)*-vertices joined by a* 2*-thread and whose* 2*-vertices of* 1*-threads are joined by a soft path.*

Proof. Let we have a shortest cycle $C = x_0v_0x_1v_1x_2 \dots v_{k+1}x_{k+2}$, where $v_1, v_2, \dots, v_k$ are (1, 1, 1)-vertices adjacent to $w_1, w_2, \dots, w_k$ outside C, respectively, v_0 and v_{k+1} are (2, 1, 1)-vertices adjacent to w_0 and w_{k+1} outside C, respectively, and x_i's are 2-vertices (see Fig. 2). Since C is shortest, all w_i's are pairwise distinct.

Take a list 2-facial coloring of $G - C$ and uncolor w_i's, where $0 \leq i \leq k + 1$. We can assume that $|A(v_i)| = 4$, $|A(w_i)| = 2$, $|A(x_i)| = |A(x_{k+2})| = 5$, where $0 \leq i \leq k + 1$.

We first put $c(x_1) \notin A(w_0)$ and then put $c(v_0) \notin A(w_0) \cup c(x_1)$. Now we can color the uncolored vertices in the following order: $w_1, v_1, x_2, w_2, v_2, x_3, \dots, w_{k+1}, v_{k+1}, x_{k+2}, x_0$, and w_0. □

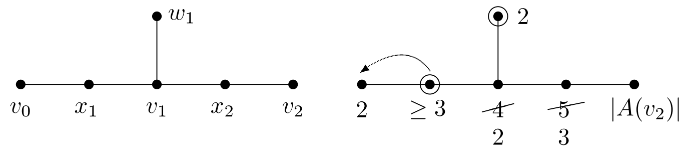

Fig. 3. Applying RE to v_1.

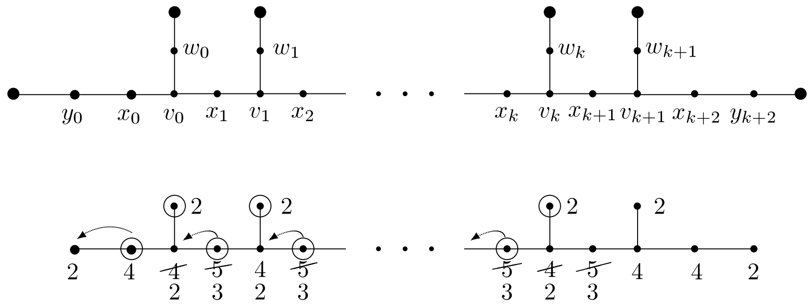

Fig. 4. The configuration in Lemma 7.

By the *operation RE* (see Fig. 3) we mean the following.

Let a vertex v_1 be incident with paths $v_1x_1v_0$, $v_1x_2v_2$, and v_1w_1, and $|A(v_0)| = |A(w_1)| = 2$, $|A(x_1)| \geq 3$, $|A(v_1)| = 4$, $|A(x_2)| = 5$. Put $c(x_1) \notin A(v_0)$, then color w_1 and delete $\{c(x_1), c(w_1)\}$ from $A(v_1)$ and $A(x_2)$.

In this notation, RE is applied to vertex v_1. Note that v_0 can be colored in the last place, while vertices v_1 and x_2 have at least 2 and 3 admissible colors, respectively.

Lemma 7. *G has no two* (2, 1, 1)*-vertices joined by a soft path.*

Proof. Suppose G has a path $y_0x_0v_0x_1v_1x_2 \dots x_{k+1}v_{k+1}x_{k+2}y_{k+2} = y_0x_0v_0(SP)v_{k+1}x_{k+2}y_{k+2}$, where $v_1, v_2, \dots, v_k$ are (1, 1, 1)-vertices, v_0 and v_{k+1} are (2, 1, 1)-vertices, y_0, y_{k+2}, and x_i's are 2-vertices. The 2-vertex adjacent to v_i, $0 \leq i \leq k+1$, and different from x_i and x_{i+1} is denoted by w_i (see Fig. 4). Note that due to Lemma 6, the 2-threads at vertices v_0 and v_{k+1} are different and that $k \geq 0$ due to Lemma 4.

Take a list 2-facial coloring of $G - SP$ and uncolor w_i, where $0 \leq i \leq k+1$, $y_0, x_0, v_0, y_{k+2}, x_{k+2}$, and v_{k+1}. We can assume that $|A(v_i)| = |A(x_0)| = |A(x_{k+2})| = 4$, $|A(w_i)| = |A(y_0)| = |A(y_{k+2})| = 2$, where $0 \leq i \leq k+1$, and $|A(x_i)| = 5$, where $0 \leq i \leq k+1$.

We first apply operation RE to v_0. More specifically, we put $c(x_0) \notin A(y_0)$, then color w_0 and delete $\{c(x_0), c(w_0)\}$ from $A(v_0)$ and $A(x_1)$. Similarly, we apply operation RE to vertices $v_1, \dots, v_k$ consecutively. As a result, $|A(v_k)| \geq 2$, $|A(x_{k+1})| \geq 3$, vertices $w_0, \dots, w_k, x_0, \dots, x_k$ are colored, while each of $y_0, v_0, \dots, v_k$ has at least two admissible colors. Now we color w_{k+1} arbitrarily and then color v_k, x_{k+1}, v_{k+1}, x_{k+2}, and y_{k+2} by Claim 2. Finally, we color $v_{k-1}, v_{k-2}, \dots, v_0$, and y_0 in this order. $\square$

Lemma 8. *G has no* (2, 1, 1)*-vertex whose* 2*-vertex of a* 1*-thread is joined by a soft path to a cycle of* (1, 1, 1)*-vertices and their common* 2*-neighbors.*

Proof. Let we have a shortest soft path $SP = x_1v_1x_2v_2 \dots x_kv_kx_{k+1}$, where x_1 is adjacent with a (2, 1, 1)-vertex v_0, while v_0 is incident with a 2-thread $v_0x_0y_0y'_0$, and $v_1, v_2, \dots, v_k$ are (1, 1, 1)-vertices adjacent to $w_1, w_2, \dots, w_k$ outside C, respectively, with the property that $x_{k+1} = w_j$, where $j < k$ (see Fig. 5). Note that $j > 0$ due to Lemma 5.

So, k is the minimum number providing a counter-example to the statement of Lemma 8. Since SP is shortest, all other w_i's are pairwise distinct whenever $0 \leq i \leq k$.

Take a list 2-facial of $G - SP$ and uncolor y_0 and w_i, where $0 \leq i \leq k$. We can assume that $|A(v_i)| = |A(x_0)| = 4$, where $0 \leq i \leq k$, $|A(w_i)| = |A(y_0)| = 2$, where $i \in \{0, \dots, k\} - j$, and $|A(x_i)| = 5$, where $1 \leq i \leq k+1$.

Put $c(x_{j+1}) \notin A(v_{j+1})$, then color w_{j+1}, and further put $c(v_{j+2}) \notin A(v_{j+1}) - c(w_{j+1})$. Now we can color the uncolored vertices in the following order: $w_{j+2}, x_{j+2}, x_{j+3}, w_{j+3}, v_{j+3}, \dots, w_k, v_k, x_{k+1}$. If $j = 1$, then $|A(x_1)| = 3$ and $|A(v_1)| = 2$; we color w_0 and apply Lemma 2. If $j \geq 2$, then we are in the situation of Lemma 7. More specifically, we apply operation RE to vertices $v_{j-1}, v_{j-2}, \dots, v_1$ consecutively and use Lemma 2. Finally, we color v_{j+1}. $\square$

2.2. Feeding paths and initial discharging

R1: Every 2-vertex that belongs to a 1-thread gets charge 1 from its end vertices, while a 2-vertex that belongs to a 2-thread gets charge 2 from the adjacent 3^+-vertex.

Note that after applying R1, the charge of every 2-vertex vanishes, while the charges of (2, 1, 1)- and (2, 2, 0)-vertices are equal to -1 and the charges of the other 3-vertices are nonnegative by Lemma 4.

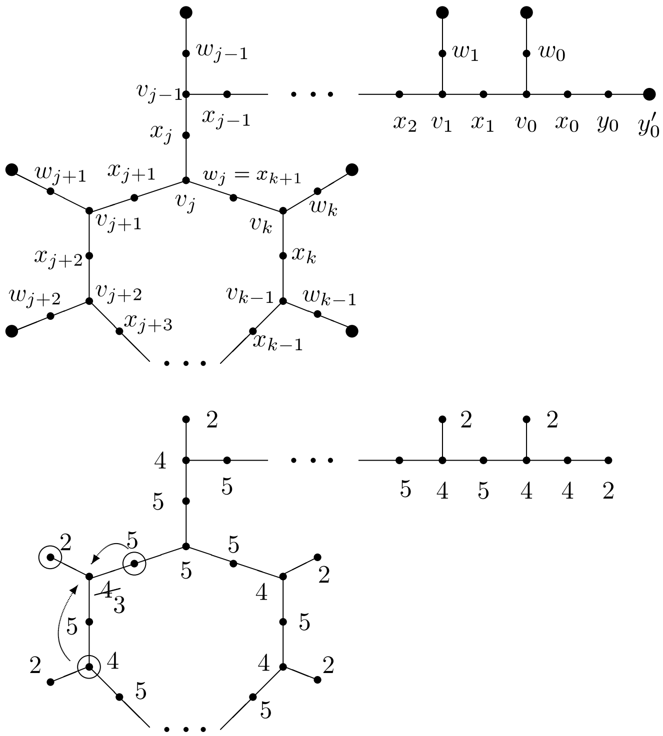

Fig. 5. The configuration in Lemma 8.

We now introduce the notion of sponsor as follows. Every (2, 1, 1)-vertex v_0 gets two *feeding paths*, FP_1 and FP_2, each of which is a soft path starting at a 2-vertex, x_1 or x_1', of a 1-thread incident with v_0 and ending in a 2-vertex, x_{k+1} or $x'_{k'+1}$, adjacent to a sponsor, v_{k+1} or $v'_{k'+1}$, respectively. Namely, $FP_1 = x_1v_1x_2 \ldots x_{k+1}$ and $FP_2 = x_1'v_1'x_2' \ldots x'_{k'+1}$, where $x_1 \neq x_1'$, while v_i and $v'_{i'}$ are (1, 1, 1)-vertices, $1 \le i \le k$, $1 \le i' \le k'$, whereas v_{k+1} and $v'_{k'+1}$ are not (1, 1, 1)-vertices; moreover, FP_1 and FP_1 are shortest paths with these properties.

Due to Lemmas 6 and 7, a feeding path of a (2, 1, 1)-vertex cannot go to another (2, 1, 1)-vertex. By Lemma 5, a feeding path of a (2, 1, 1)-vertex v_0 cannot go to v_0, i.e., $FP_1 \neq FP_2$. Finally, a feeding path cannot close on itself by Lemma 8. In view of the finiteness of our G, this means that v_{k+1} and $v'_{k'+1}$ exist. They are called *the sponsors* for the (2, 1, 1)-vertex v_0. Note that each sponsor having degree 3 is incident with at least one 0-thread.

It also follows from Lemma 7 that no two feeding paths have a (1, 1, 1)-vertex in common or are joined by a 1-thread. This implies that at most one feeding path enters any 1-thread of any sponsor, i.e., no feeding path branches. On the other hand, it is not excluded that $v_{k+1} = v'_{k'+1}$.

R2: Every (2, 1, 1)-vertex gets charge $\frac{1}{2}$ along each feeding path from its sponsors.

After applying R2, the charge of every (2, 1, 1)-vertex vanishes, the charge of every (2, 2, 0)-vertex is still -1, while the charges of all other 3-vertices are still nonnegative, except for (2, 1, 0)-vertices that are sponsors (and then have charge $-\frac{1}{2}$). Such (2, 1, 0)-vertices will be called (2, 1*, 0)-*vertices*.

Indeed, it suffices to observe that (1, 1, 1)-vertices have charge 0, while any other 3-vertex, except for a (2, 1, 1)-vertex, is incident with a 0-thread and therefore, having initial charge 3, can end up with a negative charge only if one of its incident 1-threads takes away two units of charge by R1, while the other takes $1 + \frac{1}{2}$ by R1 and R2.

2.3. *Final discharging and completing the proof of Theorem 1*

R3: Every (2, 2, 0)-vertex gets charge 1 from the adjacent 3^+-vertex.

R4: Every (2, 1*, 0)-vertex gets charge $\frac{1}{2}$ from the adjacent 3^+-vertex.

A vertex is *loaded* if it is either a (2, 2, 0)-vertex or a (2, 1*, 0)-vertex. By Lemma 11 below, applying Rules R3 and R4 does not result in a conflict.

Claim 9. *Suppose a graph H consists of path $u_2u_1uu_1'u_2'$ of vertices having* 2 , 4 , 5 , 3 , *and* 2 *admissible colors, respectively, and path $v_2v_1vv_1'$ of vertices having* 2 , 4 , 4 , *and* 2 *admissible colors, respectively, and u is adjacent to v. Then H is 2-facial colorable with admissible colors.*

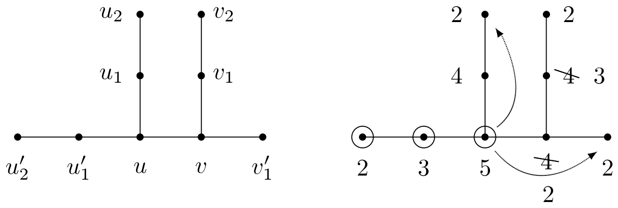

Fig. 6. The graph H in Claim 9.

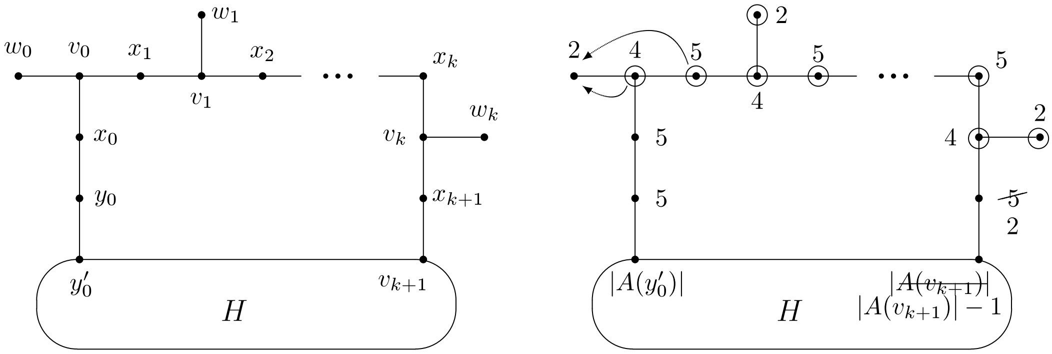

Fig. 7. The graph H^* in Claim 10.

Proof. Put $c(u) \notin A(u_2) \cup A(v_1')$. Then we color u_2' and u_1' in this order. Now v_1', v, v_1, and v_2 can be colored by Claim 1. Finally, we color u_1 and u_2 in this order (Fig. 6). $\square$

Claim 10. *Suppose a graph H^* consists of a graph H and a path $P = y_0'y_0x_0v_0x_1 \dots v_kx_{k+1}v_{k+1}$, where $P \cap H = \{y_0', v_{k+1}\}$ and v_i is adjacent to w_i for $0 \le i \le k$. Let A be a list on H^* such that $|A(w_0)| = \dots = |A(w_k)| = 2$, $|A(v_0)| = |A(v_1)| = \dots = |A(v_k)| = 4$, $|A(y_0)| = |A(x_0)| = \dots = |A(x_{k+1})| = 5$. If A contains a 2-facial coloring of $H \cup x_{k+1}$ after deleting at most one color from $A(v_{k+1})$ and at most three colors from $A(x_{k+1})$, then A contains a 2-facial coloring of H^*.*

Proof. We first put $c(x_1) \notin A(w_0)$ and then put $c(v_0) \notin A(w_0) \cup c(x_1)$. Now we color the vertices in the following order: $w_1, v_1, x_2, w_2, v_2, x_3, \dots, x_k, w_k, v_k$. Then delete $c(v_k)$ from $A(v_{k+1})$ and $A(x_{k+1})$, and delete $c(w_k)$ and $c(x_k)$ from $A(x_{k+1})$. Now x_{k+1} has at least 2 admissible colors. If $H \cup x_{k+1}$ can be 2-facial colored, then H^* is also 2-facial colorable since vertices y_0, x_0, and w_0 can be colored in last place in this order (Fig. 7). $\square$

Lemma 11. *G has no two loaded vertices adjacent to each other.*

Proof. Let u and v be adjacent loaded vertices. We must exclude the following possibilities: (a) (220−022); (b) (220−021*); (c) (21*0 − 01*2).

To prove (a) and (b), we take a 2-facial coloring of $G - v$ and uncolor u and the 2-vertices of the threads incident with u and v. Due to Claim 9, this coloring can be extended to the whole G; a contradiction.

Instead of (c), we prove the stronger statement that a $(2, 1^*, 0)$-vertex u cannot be adjacent to a $(2, 1, 0)$-vertex v. Indeed, suppose v is incident with a 2-thread $vv_1'v_2'v_3'$ and 1-thread $vv_1''v_2''$, while u is incident with a 2-thread $uu_1u_2u_3$ and a feeding path $FP(u) = x_1v_1x_2 \dots x_kv_kx_{k+1}$ going from a $(2, 1, 1)$-vertex z to u, where u is adjacent to x_{k+1}, z is adjacent to x_1 and is incident with 2-thread $zx_0y_0y_0'$, and v_i is adjacent to $w_i \notin FP(u)$ for $1 \le i \le k$. Here, we uncolor u, v, z, all vertices in $FP(u)$, their adjacent 2-vertices, and 2-vertices of 2-threads incident with u, v, and z.

We are easily done if $z = v_2''$ (note that now $z \notin \{u_3, v_3'\}$ since $g(G) \ge 12$) by first putting $c(v_1'') \notin A(x_0)$ and then coloring the other uncolored vertices in this order: $v, v_2, v_1, u, u_2, u_1, x_{k+1}, w_k, v_k, \dots, x_1, z, y_0, x_0$. (Alternatively, we can first put $c(x_1) \notin A(x_0)$ and then argue as in the next paragraph.)

So, from now on suppose that $z \ne v_2''$. If $z \notin \{u_3, v_3'\}$; then we apply RE along $FP(u)$ successively to $z, v_1, \dots, v_k$, so that the 2-vertices of $FP(u)$ are colored while $|A(z)| = |A(v_1)| = \dots = |A(v_k)| = 2$, then color the small tree T spanned by v, u and the 2-vertices of their incident 1- and 2-threads using Claim 9, and finally extend the coloring obtained of T to $v_k, \dots, v_1, z$ in this order.

If $z \in \{u_3, v_3'\}$, then we apply Claim 10. Note that in both cases, $z = u_3$ or $z = v_3'$, the residual graph $H \cup x_{k+1}$ (in the notation of Claim 10) consists of six vertices $x_{k+1}, u, v, v_1', v_2', v_1''$ (if $z = u_3$) or $x_{k+1}, u, u_1, u_2, v, v_1''$ (if $z = v_3'$), and we can assume that $|A(x_{k+1})| = |A(v_1'')| = |A(v_2')| = 2$ and $|A(u)| = |A(v)| = |A(v_1')| = 4$ or $|A(x_{k+1})| = |A(u_2)| = |A(v_1'')| = 2$ and $|A(u)| = |A(v)| = |A(u_1)| = 4$. So, $H \cup x_{k+1}$ is the neighborhood of a $(2, 2, 1)$-vertex and can be colored by Lemma 6. $\square$

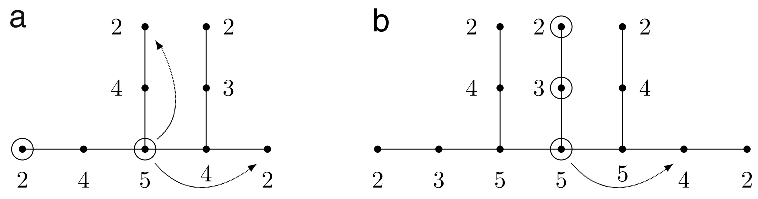

Fig. 8. Proofs in Claims 12 and 13.

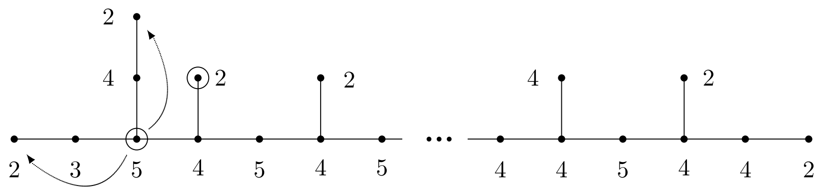

Fig. 9. Proof in Claim 14.

Claim 12. *Suppose a graph H consists of a path $u_2u_1uu_1'u_2'$ of vertices having* 2 , 4 , 5 , 4 , *and* 2 *admissible colors, respectively, and a path $v_2v_1vv_1'$ of vertices having* 2 , 3 , 4 , *and* 2 *admissible colors, respectively, where u is adjacent to v. Then H is* 2*-facial colorable with admissible colors.*

Proof. See Fig. 8(a). We put $c(u) \notin A(u_2) \cup A(v_1')$ and then color u_2'. Now v_1', v, v_1, and v_2 can be colored by Claim 1. Finally, we color u_1', u_1, and u_2 in this order. □

Claim 13. *Suppose a graph H consists of a path $u_2u_1uu_1'u_2$ of vertices having* 2 , 4 , 5 , 3 , *and* 2 *admissible colors, respectively, a path vv_1v_2 of vertices having 5,* 3 , *and* 2 *admissible colors, respectively, and a path $w_2w_1ww_1'w_2'$ of vertices having* 2 , 4 , 5 , 4 , *and* 2 *admissible colors, respectively; furthermore, v is adjacent to u and w. Then H is* 2 *-facial colorable with admissible colors.*

Proof. See Fig. 8(b). Put $c(v) \notin A(w_1')$. We color v_2 and v_1 in this order, and then color path $u_2u_1uu_1'u_2$ by Claim 2. Finally, we color w, w_2, w_1, w_2', and w_1' in this order. □

Claim 14. *Suppose a graph H consists of a path $u_2u_1uu_1'u_2'$ and a path $v_1'vx_1v_1x_2 \ldots v_{k+1}x_{k+2}y_{k+2}$, where v_i is adjacent to w_i for $1 \le i \le k+1$, while u is adjacent to v. Then every list A on H such that $|A(u_2)| = |A(u_2')| = |A(v_1')| = |A(w_1)| = \cdots = |A(w_{k+1})| = |A(y_{k+2})| = 2$, $|A(u_1')| = 3$, $|A(u_1)| = |A(v)| = |A(v_1)| = \cdots = |A(v_{k+1})| = |A(x_{k+2})| = 4$, $|A(u)| = |A(x_1)| = \cdots = |A(x_{k+1})| = 5$ contains a* 2*-facial coloring.*

Proof. Put $c(u) \notin A(u_2) \cup A(u_2')$ and then color v_1'. Now we apply operation RE to vertices $v_1, \ldots, v_k$ consecutively, then color w_{k+1} arbitrarily, and apply Claim 1 to path $v_kx_{k+1}v_{k+1}x_{k+2}y_{k+2}$. Finally, we color $v_{k-1}, \ldots, v_1, v, u_1', u_2', u_1$, and u_2 in this order (Fig. 9). □

A vertex v is *overloaded* if its charge becomes negative after applying R3 and R4. Note that v might be of one of the following types: (210), (1*1*0), (11*0), (200), and (1*00), where 1* denotes a 1-thread extended by the feeding path. To complete the proof of Theorem 1, it suffices to prove that overloaded vertices cannot exist.

In what follows, if v is a (2, 1, 0)-, (1*, 1*, 0)- or (1, 1*, 0)-vertex, then by $vv_1v_2v_3$ and $vv_1'v_2'$ denote the 2- and 1-threads incident with v. If v is a (2, 0, 0)- or (1*, 0, 0)-vertex, then by $vv_1v_2v_3$ or vv_1v_2 denote the 2- or 1-thread incident with v, respectively. By $FP(v)$ denote the feeding path ending at v.

By u or w denote the loaded vertices adjacent to v, while $uu_1u_2u_3$, $uu_1'u_2'$ and $ww_1w_2w_3$, $ww_1'w_2'$ are the 2- and 1-threads incident with u and w, respectively.

Lemma 15. *G has no overloaded vertex.*

Proof. Let v be an overloaded vertex. We have the following possibilities for v to be adjacent to (2, 2, 0)- and (2, 1*, 0)-vertices:

(A) (220 — 012);
(B) (21*0 — 012);
(C) (220 — 01*1*);
(D) (21*0 — 01*1*);
(E) (220 — 011*);
(F) (220 — 020 — 022);
(G) (220 — 020 — 021*);
(H) (220 — 01*0 — 022).

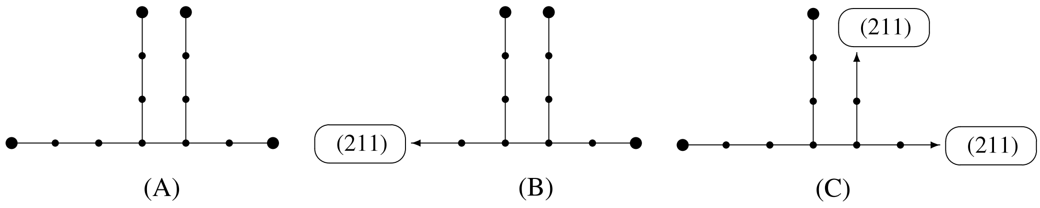

Fig. 10. Cases A, B, and C in Lemma 15.

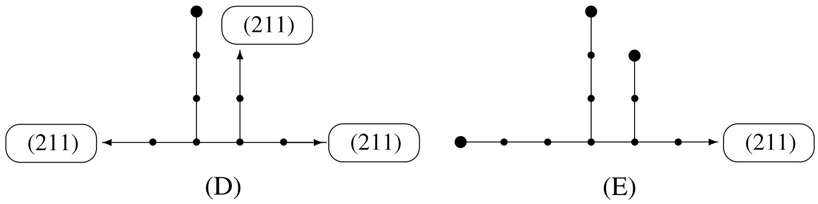

Fig. 11. Cases D and E in Lemma 15.

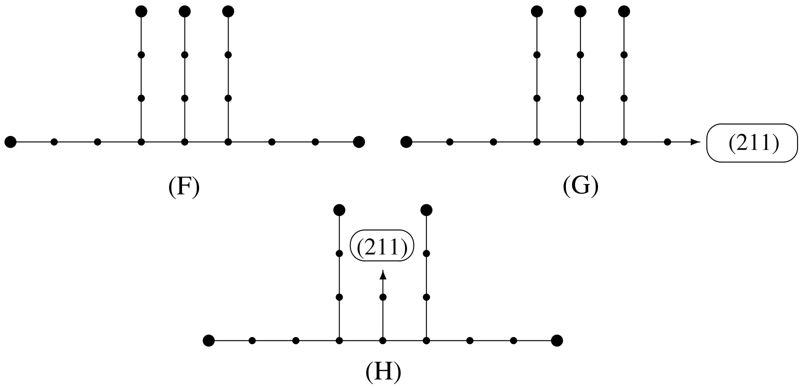

Fig. 12. Cases F, G, and H in Lemma 15.

See Figs. 10–12, where 1^*-thread is depicted by a vector directed to a (2, 1, 1)-vertex surrounded by an oval.

In fact, we have proved (A) and (B) in the course of proving Lemma 11. Since (E) is stronger than (C), we skip (C).

In proving (D), we say that two feeding paths *close* each other either if they start at the same (2, 1, 1)-vertex or if they start at two different (2, 1, 1)-vertices joined by a 2-thread. Let u be a $(2, 1^*, 0)$-vertex adjacent to a $(1^*, 1^*, 0)$-vertex v, and let u be incident with a feeding path P_1, while v be incident with a feeding paths P_2 and P_3. By the symmetry between P_2 and P_3, we can assume that if there is a closing between P_1, P_2, and P_3, then P_3 participates in closing (so, P_2 never closes with P_1).

Take a 2-facial coloring of $G - v$ and uncolor u, u_1, u_2, v_1' (where $v_1' \in P_3$) and all vertices of P_1 and P_2 (including their initial (2, 1, 1)-vertices) and the 2-vertices of their incident 1- and 2-threads. Now our proof splits. If neither P_1 nor P_2 is a feeding path for u_3, then we are done by applying Claim 14. If P_1 or P_2 is a feeding path for u_3, then by applying Claim 10 we get into the situation resolved in Lemma 7.

We can prove (E) in the same fashion as we proved (c) in Lemma 11. If there is no closing, then the proof follows easily from Claim 12. Otherwise, we apply Claim 10. Here, the graph $H \cup x_{k+1}$ consists of a path $u_2u_1uvx_{k+1}$ augmented by edge vv_1, where $|A(u_2)| = |A(v_1)| = |A(x_{k+1})| = 2$, $|A(u_1)| = 4$, $|A(u)| = 5$, $|A(v)| = 3$. Thus, we color x_{k+1}, v_1, v, u, u_2, and u_1 in this order.

The proof of (F) follows immediately from Claim 13. It remains to prove (G) and (H). Note that in both cases there is a unique feeding path P. If P does not create a subgraph H^* described in Claim 10 (i.e., there is no closing), then we apply operation RE to vertices of P as we did before and use Claim 13 again. Otherwise, we have one of the following types of subgraph H in Claim 10: $(220 - 022)$, $(221 - 121)$, and $(320 - 021)$ in (G) and $(220 - 013)$ in (H). The colorability of all of them but the second follows from (A) of this lemma, while the colorability of $(221 - 121)$ follows from the proof of Lemma 7. Now H^* can be colored by Claim 10, a contradiction. □

So, after discharging according to R1–R4, the charges of all 3-vertices are nonnegative. It remains to observe that every 4^+-vertex v has $\mu^*(v) \geq 5d(v) - 12 - 2d(v) \geq 0$, since v sends at most 2 along every incident edge. This contradicts (2) and thus completes the proof of Theorem 1.

Acknowledgments

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References

[1] K. Appel, W. Haken, Every planar map is four colorable, J. Recreational Math. 9 (3) (1976/77) 161–169.
[2] O.V. Borodin, Solution of the Ringel problem on vertex-face coloring of planar graphs and coloring of 1-planar graphs, Metody Diskret. Analiz. 41 (1984) 12–26 (in Russian).
[3] O.V. Borodin, New proof of six color theorem, J. Graph Theory 19 (1995) 507–521.
[4] O.V. Borodin, A.N. Glebov, A.O. Ivanova, T.K. Neustroeva, V.A. Tashkinov, Sufficient conditions for the 2-distance $\Delta + 1$-colorability of plane graphs, Sib. Elektron. Mat. Izv. 1 (2004) 129–141 (in Russian).
[5] O.V. Borodin, S.G. Hartke, A.O. Ivanova, A.V. Kostochka, D.B. West, (5, 2)-coloring of sparse graphs, Sib. Elektron. Mat. Izv. 5 (2008) 417–426.
[6] O.V. Borodin, A.O. Ivanova, 2-Distance $(\Delta + 2)$-coloring of planar graphs with girth six and $\Delta \geq 18$, Discrete Math. 309 (2009) 6496–6502.
[7] O.V. Borodin, A.O. Ivanova, List 2-distance $(\Delta + 2)$-coloring of planar graphs with girth six, Europ. J. Combin. 30 (2009) 1257–1262.
[8] O.V. Borodin, A.O. Ivanova, 2-distance 4-colorability of planar subcubic graphs with girth at least 22, Discuss. Math. Graph Theory (in press).
[9] O.V. Borodin, A.O. Ivanova, Acyclic 3-choosability of planar graphs with no cycles of length from 4 to 11, Sib. Elektron. Mat. Izv. 7 (2010) 275–283.
[10] O.V. Borodin, A.O. Ivanova, A.V. Kostochka, Oriented vertex 5-coloring of sparse graphs, Diskretn. Anal. Issled. Oper. 13 (1) (2006) 16–32. in Russian.
[11] O.V. Borodin, A.O. Ivanova, T.K. Neustroeva, 2-distance coloring of sparse plane graphs, Sib. Elektron. Mat. Izv. 1 (2004) 76–90 (in Russian).
[12] O.V. Borodin, A.O. Ivanova, T.K. Neustroeva, List 2-distance $(\Delta + 1)$-coloring of planar graphs with given girth, Diskretn. Anal. Issled. Oper. 14 (3) (2007) 13–30 (in Russian).
[13] O.V. Borodin, A.O. Ivanova, T.K. Neustroeva, Sufficient conditions for 2-distance $(\Delta + 1)$-colorability of planar graphs of girth 6, Diskretn. Anal. Issled. Oper. 12 (3) (2005) 32–47 (in Russian).
[14] O.V. Borodin, A.O. Ivanova, T.K. Neustroeva, Sufficient conditions for the minimum 2-distance colorability of planar graphs with girth 6, Sib. Elektron. Mat. Izv. 3 (2006) 441–450 (in Russian).
[15] O.V. Borodin, D. Sanders, Y. Zhao, On cyclic colorings and their generalizations, Discrete Math. 203 (1999) 23–40.
[16] Z. Dvořák, D. Kràl, P. Nejedlỳ, R. Škrekovski, Coloring squares of planar graphs with girth six, European J. Comb. 29 (4) (2008) 838–849.
[17] Z. Dvořák, R. Škrekovski, M. Tancer, List-coloring squares of sparse subcubic graphs, SIAM J. Discrete Math. 22 (1) (2008) 139–159.
[18] F. Havet, Choosability of square of planar subcubic graphs with large girth, Discrete Math. 309 (2009) 3353–3563.
[19] A.O. Ivanova, 2-facial 4-colorability of plane graphs with girth at least 22, Mathematical Notes of Yakutsk University (in Russian) (in press).
[20] A.O. Ivanova, List 2-distance $(\Delta + 1)$-coloring of planar graphs with girth at least 7, Diskretn. Anal. Issled. Oper. 17 (5) (2010) 22–36 (in Russian).
[21] A.O. Ivanova, A.S. Solov'eva, 2-Distance $(\Delta + 2)$-coloring of sparse planar graphs with $\Delta = 3$, Math. Notes Yakutsk University 16 (2) (2009) 32–41 (in Russian).
[22] T. Jensen, B. Toft, Graph Coloring Problems, John Willey & Sons, New York, 1995.
[23] D. Kràl, T. Madaras, R. Škrekovski, Cyclic, diagonal and facial colorings, European J. Combin. 26 (2005) 473–490. 3–4.
[24] M. Montassier, A. Raspaud, A note on 2-facial coloring of plane graphs, Inform. Process. Lett. 98 (6) (2006) 235–241.
[25] O. Ore, M.D. Plummer, Cyclic coloration of plane graphs, in: Recent Progress in Combinatorics (Proc. Third Waterloo Conf. on Combinatorics, 1968), Academic Press, New York, 1969, pp. 287–293.